Convex Optimization Problems

Proof of LEP 1-2 [I]

Optimization program: (notation $f_0(x) = f(x)$)

$$(P) \quad \inf_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{array}{ll} f_i(x) \le 0 & i = 1 \dots p \\ f_j(x) = 0 & j = p+1 \dots m \\ x \in C & \end{array}$$

DEF: $S := \{x \in C \mid f_i(x) \le 0, f_j(x) = 0, i = 1 \dots p, j = p+1 \dots m\}$

ASSUMPTIONS:

(A1) $C \subseteq \mathbb{R}^n$ convex s.t. $\operatorname{supp} f_i \supseteq C$, $i = 1 \dots m$

(A2) $f_i: \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ convex for $i = 1 \dots p$

(A3) $f_j: \mathbb{R}^n \to \mathbb{R}$ affine for $j = p+1 \dots m$

REM: (A1)-(A3) $\Rightarrow$ S convex
and furthermore if C closed $\wedge$ $f_i|_C$, $f_0|_C$ continuous (implied by convexity) $\Rightarrow$ S closed

ASSUMPTIONS:

(A4) for convex S holds:

a) $\exists \bar{x} \in S \cap C^\circ$

b) $\forall f_i$, $i = 1 \dots p$, not affine $\exists x_i \in S$: $f_i(x_i) < 0$

This is called Slater condition.

THM: (P) fulfilling
- (A1)-(A3)
- (A4) Slater cond
- $\alpha := \inf\{f(x) \mid x \in S\} \in \mathbb{R}$

$\Rightarrow \exists y \in \mathbb{R}^m$, $y_i \ge 0$ $i = 1 \dots p$ s.t.

$$f(x) + \sum_{i=1}^{p} y_i f_i(x) \ge \alpha \quad \text{for all } x \in C$$

REM: Only finiteness of α is assumed not existence of $x^* \in C$ with $f(x^*) = \alpha$, however, if $x^* \subset C$ is the optimal value $f(x^*) = \alpha$ holds.

Proof:
- Preliminary: (A4) b) $\Rightarrow \exists \hat{x} \in S$ s.t. $f_i(\hat{x}) < 0$ for all non-affine functions, say, $i = 1 \dots \ell$

  Proof: $\forall i = 1 \dots \ell$ $\exists x_i \in S$ with $f_i(x_i) < 0$, $f_k(x_i) \le 0$ and $f_j(x_i) = 0$ for $k \ne i$, $k = 1 \dots p$

  $\hat{x} := \frac{1}{\ell}(x_1 + x_2 + \dots x_n) \in S$ due to convexity

  and $f_i(\hat{x}) = f_i\left(\frac{1}{\ell} \sum_{n=1}^{\ell} x_n\right)$
  $$\le \frac{1}{\ell} \sum_{i=n}^{\ell} f_i(x_n) = \frac{1}{\ell}\Big(\underbrace{f_i(x_1)}_{\le 0} + \dots + \underbrace{f_i(x_i)}_{< 0} \dots \underbrace{f_i(x_\ell)}_{\le 0}\Big)$$
  < 0 $\square$

- To simplify the cases, let us assume $\exists \hat{x} \in S$ with $f_i(x_i) < 0$ for $i = 1 \dots p$ (also for the affine function)

(*)
- By (A4) a) $\exists \bar{x} \in S \cap C^\circ$ so that by convexity $\bar{x}$ can be ~~selected~~ to fulfill $f_i(\bar{x}) < 0$ $i = 1 \dots p$

- WLOG $\alpha = 0$

PART 1: Without (A4) we prove that $\exists z \in \mathbb{R}^{m+1}$, $z \ne 0$:

$$z_i \ge 0, \ i = 1 \dots p \ \wedge \ \sum_{i=0}^{p} z_i f(x) \ge 0 \quad \forall x \in C$$

PART 2: With (A4) we show that $z_0 > 0$

$\Rightarrow y := (z_i / z_0)_{i = 1 \dots m}$

Proof of Part 1:

Proof of Lemma 1-2

$$A := \left\{ v \in \mathbb{R}^{m+1} \mid \exists x \in C : \begin{array}{l} v_0 > f_0(x) \\ v_i \geq f_i(x) \quad i=1..p \\ v_j = f_j(x) \quad j=p+1...m \end{array} \right\}$$

C convex $\wedge$ f_i convex $i=1..p$ $\wedge$ f_j affine $j=p+1..m$
$\Leftarrow$ f_j convex

$\Rightarrow$ • A convex
• $0 \notin A$ because of $\alpha = 0$
• $A \neq \emptyset$ as v_i can be chosen arbitrarily negative

LEMMA: $A \neq \emptyset$, $0 \notin A$, A convex $\subseteq \mathbb{R}^{m+1}$
$\Rightarrow \exists z \in \mathbb{R}^{m+1}, z \neq 0$ s.t. i) $z \cdot v \geq 0 \quad \forall v \in A$
ii) $\exists \bar{v} \in A : z \cdot \bar{v} > 0$

• But $w \in A \Rightarrow v + \lambda w \in A \quad \forall \lambda \geq 0, w \geq 0$
and $\lim_{\lambda \to +\infty} z \cdot (v + \lambda w) \geq 0 \Rightarrow z \cdot w \geq 0 \Rightarrow z \geq 0$

• for all $\begin{array}{c} x \in C \\ \varepsilon > 0 \end{array}$ $\exists \; v_\varepsilon := \begin{pmatrix} f_0(x) + \varepsilon \\ f_i(x) \\ f_j(x) \end{pmatrix} \in A$

$0 \leq z \cdot v_\varepsilon = z_0 (f(x) + \varepsilon) + \sum_{i=1}^{p} z_i f_i(x)$

$\varepsilon \to 0 \Rightarrow \quad 0 \leq z_0 f(x) + \sum_{i=1}^{p} z_i f_i(x) \quad , z_i \geq 0, z \neq 0$

Proof of Part 2:

• Let us assume $z_0 = 0$

$\Rightarrow \quad v = (f(\hat{x}) + 1, f_{i=1..p}(\hat{x}), 0, \ldots, 0) \in A$

because of (A^4) a) (see $*$)

$\Rightarrow \quad z \cdot v \geq 0$
but since $z_0 = 0$, $f_i(\hat{x}) < 0$
$\Rightarrow z_0, z_1 \ldots z_p = 0$
$\Rightarrow$ due to A: $\sum_{j=p+1}^{m} z_j f_j(x) \geq 0 \quad \forall x \in C$

and since $\{0\}$ and A well separated $\exists \; \tilde{x} \in C$:
$$\sum_{j=p+1}^{m} z_j f_j(\tilde{x}) > 0$$

• because $\hat{x} \in C^\circ$ we have $\hat{x} - \varepsilon(\tilde{x} - \hat{x}) \in C$

• since f_j affine for $j = p+1 \ldots m$

$$f_j(\hat{x} - \varepsilon(\tilde{x} - \hat{x})) = \underbrace{f_j(\hat{x})}_{=0} - \varepsilon [f_j(\tilde{x}) - f_j(\hat{x})]$$
$$= -\varepsilon f_j(\tilde{x})$$

$$\Rightarrow \sum_{j=p+1}^{m} z_j f_j(\hat{x}) = -\varepsilon \sum_{j=p+1}^{m} z_j f_j(\tilde{x}) < 0 \quad ↯$$

$\Rightarrow z_0 \neq 0$ but $z_0 > 0$. $\square$